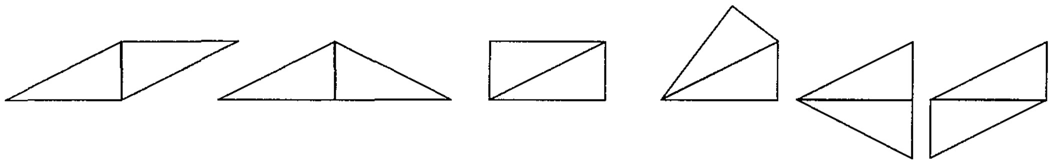
UK Junior Mathematical Olympiad 2002 Solutions

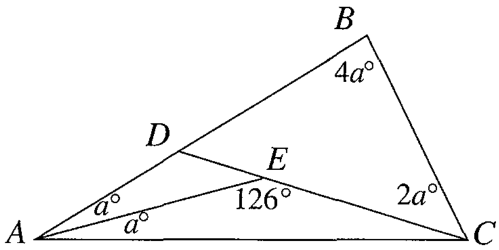
A1 **6** The diagrams show all possibilities.



A2 **0** $[(-1) + (-1)^2] \div [(-1) - (-1)^2] = 0 \div (-2) = 0.$

A3 **$3\pi r + 2r$** The length of each arc is one quarter of the perimeter of a circle of radius r, i.e. $\frac{1}{2}\pi r$. Therefore the perimeter $= 6 \times \frac{1}{2}\pi r + 2r = 3\pi r + 2r$.

A4 **18** The exterior angle of a triangle is equal to the sum of the two interior opposite angles. Applying this theorem to ΔBCD:



$$\angle ADE = \angle DBC + \angle BCD = 4a^\circ + 2a^\circ = 6a^\circ.$$

Applying the same theorem to ΔADE:

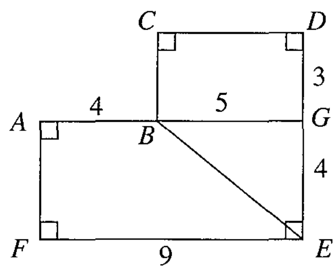
$\angle AEC = \angle EAD + \angle ADE$; therefore $126^\circ = a^\circ + 6a^\circ$ and so $a = 126 \div 7 = 18$.

A5 **280** The mean of the seven numbers $= 2002 \div 7 = 286$. This is the fourth of the numbers, so the smallest of them is $286 - 6 = 280$.

A6 **16, 625** $10\,000 = 10^4 = 2^4 \times 5^4 = 16 \times 625$. This is the only correct answer. If 10 000 is written as the product of two other factors then at least one of these will be a multiple of 2 and 5, which means that it ends in the digit zero.

A7 **$\frac{27}{25}$** $\frac{2}{3} \times \frac{5}{6} \times X = \frac{3}{4} \times \frac{4}{5} \times Y$, i.e. $\frac{5X}{9} = \frac{3Y}{5}$, i.e. $25X = 27Y$, i.e. $\frac{X}{Y} = \frac{27}{25}$.

A8 **$\sqrt{41}$** Applying Pythagoras' Theorem to ΔBGE:



$$BE^2 = BG^2 + GE^2 = 5^2 + 4^2 = 25 + 16 = 41.$$

A9 **$\frac{4}{9}$** $x + 6 = \frac{1}{4}$ so $x + 8 = 2\frac{1}{4} = \frac{9}{4}$. Therefore $\frac{1}{x+8} = \frac{4}{9}$.

A10 **£24000** Let the three jewels in box P have a total value of £p and those in box Q have total value £q. Before the jewel is transferred:

Average value of jewels in P = £$p/3$. Average value of jewels in box Q = £$q/3$.

After the jewel is transferred:

Average value of jewels in P = £$(p - 5000)/2$.

Average value of jewels in box Q = £$(q + 5000)/4$.

Therefore: $(p - 5000)/2 = p/3 + 1000$, i.e. $3p - 15000 = 2p + 6000$, i.e. $p = 21000$;

and $(q + 5000)/4 = q/3 + 1000$, i.e. $3q + 15000 = 4q + 12000$, i.e. $q = 3000$.

The total value of all six jewels = £$(p + q)$ = £24000.

B1 (i) Let T_n denote the nth triangular number i.e. $T_n = 1 + 2 + 3 + 4 + \ldots + n$.

The smallest ascending number between 1000 and 2000 is 1234.

There are 6 ascending numbers of the form $123r$ (1234, 1235, 1236, 1237, 1238 and 1239).

There are 5 ascending numbers of the form $124r$ (1245, 1246, 1247, 1248 and 1249).

Similarly, there are 4 ascending numbers of the form $125r$, 3 ascending numbers of the form $126r$, 2 ascending numbers of the form $127r$ and just 1 number of the form $128r$.

So the number of ascending numbers of the form $12qr$ is $1 + 2 + 3 + 4 + 5 + 6 = T_6 = 21$.

In the same way, we may show that;

the number of ascending numbers of the form $13qr$ is $1 + 2 + 3 + 4 + 5 = T_5 = 15$;

the number of ascending numbers of the form $14qr$ is $1 + 2 + 3 + 4 = T_4 = 10$;

the number of ascending numbers of the form $15qr$ is $1 + 2 + 3 = T_3 = 6$;

the number of ascending numbers of the form $16qr$ is $1 + 2 = T_2 = 3$;

the number of ascending numbers of the form $17qr$ is 1, i.e. 1789, which is the largest ascending number between 1000 and 2000.

So the number of ascending numbers between 1000 and 2000 is $21 + 15 + 10 + 6 + 3 + 1 = 56$.

(ii) We may use a similar method to show that:

the number of ascending numbers between 2000 and 3000 is $15 + 10 + 6 + 3 + 1 = 35$; between 3000 and 4000 is $10 + 6 + 3 + 1 = 20$; between 4000 and 5000 is $6 + 3 + 1 = 10$; between 5000 and 6000 is $3 + 1 = 4$; between 6000 and 7000 is 1, i.e. 6789, which is the largest ascending number between 1000 and 10000.

So the number of ascending numbers between 1000 and 10000 is $56 + 35 + 20 + 10 + 4 + 1$ which is 126.

Alternative solution

(i) Begin by considering all four digit numbers of the form $1pqr$ which are not necessarily ascending, but in which all four digits are different and non-zero. There are 8 choices for p, then 7 for q, finally 6 for r. Therefore there are $8 \times 7 \times 6$ such numbers in all. Six at a time, these have the same digits: $1pqr$, $1prq$, $1qpr$, $1qrp$, $1rpq$, $1rqp$. Of any such set of six, exactly one is ascending. Therefore the number of ascending numbers between 1000 and 2000 is $8 \times 7 \times 6 \div 6 = 56$.

(ii) Similarly, consider all four digit numbers of the form $pqrs$ which are not necessarily ascending, but in which all four digits are different and non-zero. There are $9 \times 8 \times 7 \times 6$ such numbers in all. Each particular combination of four digits will occur twenty-four times and of any such set of twenty-four, exactly one is ascending. Therefore the number of ascending numbers between 1000 and 10000 is $9 \times 8 \times 7 \times 6 \div 24 = 126$.

B2 (i) Each of the five teams plays four matches, so the total number of matches played is $(5 \times 4) \div 2 = 10$.

A drawn match results in a total of two points being awarded, whereas a match which is not drawn results in a total of three points being awarded. So the total number of points awarded in the 10 matches differs from its maximum possible value by the number of drawn matches.

For 10 matches, the maximum number of points awarded is $3 \times 10 = 30$, but the total number of points awarded were $10 + 9 + 4 + 3 + 1 = 27$. So there were three drawn matches.

(ii) Looking at the total points won by the teams we may deduce that:

Yellows won 3 matches and drew 1 match;

Reds won 3 matches and lost 1 match (if they had drawn any matches they must have drawn at least 3 since their points total is a multiple of 3, but this would have required them to play at least 5 matches);

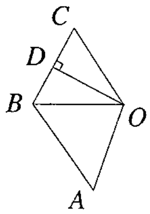
Greens won 1 match, drew 1 match and lost 2 matches (they could not have won 4 points by drawing all four matches since only 3 matches in total were drawn);
Pinks drew 1 match and lost 3 matches.

We may now deduce that Blues drew 3 matches and lost 1 match since 3 matches in total were drawn and, of the other teams, 3 teams drew 1 match each and the other did not draw any matches. Therefore Blues drew with Yellows, Greens and Pinks.

As Yellows drew with Blues, they must have won their other three games. Therefore the match which Reds lost was that against Yellows and they won their other three matches, so Greens lost to both Yellows and Reds. The match which Pinks drew was against Blues and as they lost their other three matches, we deduce that Greens beat Pinks.

In summary, Greens drew with Blues, lost to Yellows, lost to Reds and beat Pinks.

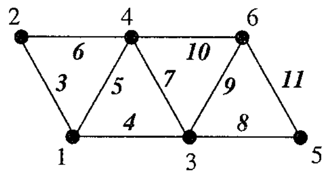
B3 In triangles OCB and OAB: $CB = AB$ (given); $OC = OA$ (radii of the circle) and OB is common to both triangles. So triangles OCB and OAB are congruent (SSS) and the area of triangle OBC is 60 cm^2.



Let D be the midpoint of BC. As triangle OBC is isosceles, OD is perpendicular to BC. Area of triangle $OBC = \frac{1}{2}BC \times OD = 60\text{ cm}^2$ and so $OD = (60 \div 5)\text{ cm} = 12\text{ cm}$.

Applying Pythagoras' Theorem to triangle OBD: $OB^2 = BD^2 + OD^2 = 5^2 + 12^2 = 169 = 13^2$. Therefore the radius of the circle is 13 cm.

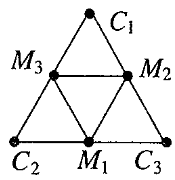
B4 (i) The task is possible to do for Network A, as shown.



(ii) It is not possible, however, for Network B. In proving this, we call the sum of the two numbers at the ends of an edge the 'weight' of that edge. Note that three of the nodes are connected to exactly two nodes, while the other three are connected to exactly four nodes. Thus the total of the nine weights must always be an even number, whichever numbers are placed at the nodes.

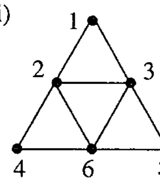
Now the smallest possible weight is 1 + 2 = 3, while the largest possible weight is 5 + 6 = 11. As there are exactly nine edges, we deduce that for the weights of each edge to be different they must take the values 3, 4, 5, 6, 7, 8, 9, 10 and 11. However, the total of these is 63, an odd number, so the task is impossible.

Alternative proof.



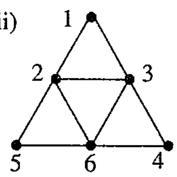
We note from the above proof that the weights of the 9 edges must be 3, 4, ... 11 and may deduce that 1 and 2 must be at the ends of the same edge (i.e. 'connected') to give a weight of 3, 1 and 3 must be connected to give a weight of 4 and, similarly, 6 must be connected to both 4 and 5 to give weights of 10 and 11 respectively.

(i)



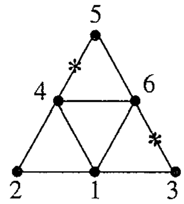
We denote the nodes at the corners of the large triangle as C_1, C_2 and C_3 and those at the midpoints of the sides of this triangle as M_1, M_2 and M_3.

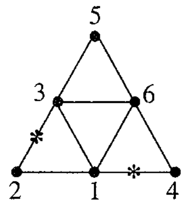
(ii)



Consider the networks for which 1 is placed at a corner node, e.g. C_1. (The symmetry of the network means that if the task is not possible with 1 placed at one particular corner, it will not be possible if it is placed at either of the other corners.) Then 2 and 3 must be placed at M_2 and M_3, or vice versa, and the symmetry of the network means we have to consider only one of these two cases. For 6 to be connected to both 4 and 5, it must now be placed at M_1, but diagrams (i) and (ii) show that the two networks corresponding to this both fail to satisfy the original condition. In (i), two of the weights are 8 while in (ii), two of the weights are 7. Thus, the task is impossible if 1 is placed at a corner node.

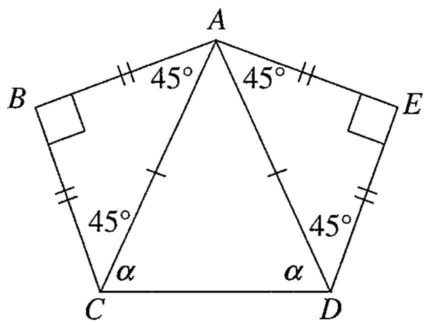
Suppose the task was possible with 6 at a corner node. Then, replacing each number u with $7 - u$, a solution with 1 at a corner node would be obtained. This has been shown to be impossible, however, so we are left with the possibility that 1 and 6 both appear at mid-points. By symmetry, suppose that they appear at M_1 and M_2 respectively. If 2 and 3 are not connected, then 1 must be connected to 4 (as well as 2 and 3) to give a weight of 5. Similarly, if 4 and 5 are not connected, 6 must be connected to 3 (as well as 4 and 5) to give a weight of 9. Furthermore, as 1 and 6 are connected, 3 cannot be connected to 4, nor 2 to 5. We now have just two cases to consider. Both, however, fail to satisfy the original condition.



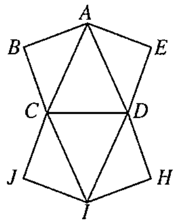


Having examined all possible cases, we may conclude that the given task is impossible.

B5

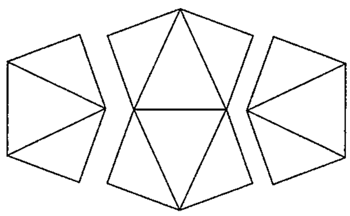


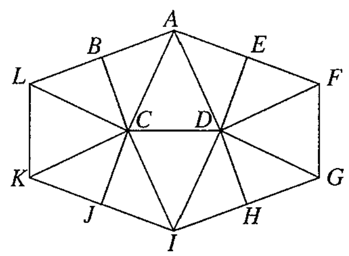
Let $\angle ACD = \angle ADC = \alpha$.
Then $\angle CAD = 180° - 2\alpha$ and
$\angle BAE = 45° + 180° - 2\alpha + 45°$
$= 270° - 2\alpha$.
Also, $\angle CDE = \alpha + 45°$.



To form a hexagon, fit two of the identical pentagons together as shown. We see that $\angle EDH = 360° - 2 \times (\alpha + 45°)$
$= 270° - 2\alpha = \angle BAE$.

Therefore two more of the identical pentagons will fit together with these to make a hexagon as shown. Note that AEF is a straight line as angles AED and DEF are both right angles and, similarly, GHI, IJK and LBA are also straight lines.





B6 The winning strategy in this game is, on your turn, to make the gap between A and B the same number of squares (possibly 0) as the gap between C and D. Therefore, in the position shown, you should move A two squares to the right, or move D two squares to the right. Your opponent must now move one of the counters so that the two gaps will be different and on your subsequent turns it will always be possible to make the two gaps the same. As the game continues, D will, sooner or later, be moved to the extreme right square and, on a subsequent move, C will be moved to the last square it can occupy i.e. the second square from the right. If your opponent moves C to this position, then you will move A to the square immediately to the left of B so that both gaps are now zero. Alternatively, you may move C to its final position yourself if D occupies the last square and your opponent places A on the square adjacent to B. In either case, your opponent is faced with a situation in which B must be moved at least one square. On your turn, you move A the same number of squares to once again reduce the gap to zero. Eventually, your opponent must move B to the square adjacent to C and you then win the game by moving A to its final position.